

A Note on Radius of Starlikeness and Convexity of p - Valent Analytic Functions

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Abstract. Let \mathcal{A}_p denote the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and Ω denote the class of functions such that $\omega(0) = 0$ and $|\omega(z)| < 1$. Let $\mathcal{P}(A, B, p, \alpha)$ be the class of functions of the form

$$p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1$$

where $\gamma = (p - \alpha)A + \alpha B$. In this paper, we define the class $\mathcal{S}_q(A, B, p, \alpha)$ of functions $f(z) \in \mathcal{A}_p$ such that

$$q + \frac{z f^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in \mathcal{P}(A, B, p, \alpha)$$

and radius of starlikeness and convexity of functions in this class are studied.

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1. INTRODUCTION

Let \mathcal{A}_p denote the class of analytic functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$. Let Ω denote the class of bounded analytic functions $\omega(z)$ in \mathcal{U} satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| \leq 1$ ($z \in \mathcal{U}$).

For functions $g(z)$ and $G(z)$ analytic in \mathcal{U} , we say that $g(z)$ is subordinate to $G(z)$ if there exists a Schwarz function $\omega(z) \in \Omega$ such that $g(z) = G(\omega(z))$. If $G(z)$ is univalent in \mathcal{U} , then $g(z)$ is subordinate to $G(z)$ if and only if $g(0) = G(0)$ and $g(\mathcal{U}) \subset G(\mathcal{U})$.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, $\mathcal{P}(A, B, p, \alpha)$ [1] denote the class of analytic functions defined in \mathcal{U} such that

$$(1.2) \quad p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1.$$

where $\gamma = (p - \alpha)A + \alpha B$. Further, $p(z) \in \mathcal{P}(A, B, p, \alpha)$ if and only if

$$(1.3) \quad p(z) = (p - \alpha)p_1(z) + \alpha, \quad p_1(z) \in \mathcal{P}(A, B)$$

where $\mathcal{P}(A, B)$ [3] is the Janowski class of functions $p_1(z)$ which are of the form

$$(1.4) \quad p_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

and are analytic in \mathcal{U} , such that $p_1(z) \in \mathcal{P}(A, B)$ if and only if

$$(1.5) \quad p_1(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1, \quad \omega(z) \in \Omega, \quad z \in \mathcal{U}.$$

We define the class $\mathcal{S}_q(A, B, p, \alpha)$ of functions $f(z) \in \mathcal{A}_p$ such that

$$(1.6) \quad q + \frac{z f^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in \mathcal{P}(A, B, p, \alpha)$$

2. MAIN RESULTS

Lemma 2.1. (Jack's Lemma) [2]: Let $\omega(z)$ be a regular function in the unit disc \mathcal{U} with $\omega(0) = 0$, then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_1 , we can write $z_1\omega'(z_1) = k\omega(z_1)$, where k is real and $k \geq 1$.

Lemma 2.2. The function

$$(2.1) \quad \omega = \begin{cases} \frac{p + \gamma z}{1 + Bz} & \text{for } B \neq 0, \\ p + \gamma z & \text{for } B = 0 \end{cases}$$

maps $|z| = r$ onto a disc centered at $C(r)$ and having the radius $\rho(r)$ given by

$$(2.2) \quad C(r) = \begin{cases} \left(\frac{p - \gamma Br^2}{1 - B^2 r^2}, 0 \right) & \text{for } B \neq 0, \\ (p, 0) & \text{for } B = 0 \end{cases}$$

and

$$(2.3) \quad \rho(r) = \begin{cases} \frac{(\gamma - pB)r}{1 - B^2 r^2} & \text{for } B \neq 0, \\ |\gamma|r & \text{for } B = 0. \end{cases}$$

Proof. Consider

$$(2.4) \quad \begin{cases} \omega = \frac{p + \gamma z}{1 + Bz} \Leftrightarrow z = \frac{\omega - p}{\gamma - B\omega} \Leftrightarrow |z|^2 = r^2 = \frac{|\omega - p|^2}{|\gamma - B\omega|^2} & \text{for } B \neq 0 \\ \Rightarrow u^2 + v^2 + \left(\frac{2\gamma Br^2 - 2p}{1 - B^2 r^2} \right) u + \frac{p^2 - \gamma^2 r^2}{1 - B^2 r^2} = 0, & \text{for } B \neq 0. \\ \omega = 1 + \gamma z \Leftrightarrow z = \frac{\omega - p}{\gamma} \Leftrightarrow |z|^2 = r^2 = \frac{|\omega - p|^2}{|\gamma|^2} & \text{for } B = 0 \\ \Rightarrow u^2 + v^2 - 2u + p^2 - \gamma^2 r^2 = 0, & \text{for } B = 0. \end{cases}$$

Lemma follows from (2.4). □

Lemma 2.3. The function

$$(2.5) \quad \omega = \begin{cases} \frac{(\gamma - pB)z}{1 + Bz} & \text{for } B \neq 0, \\ \gamma z & \text{for } B = 0 \end{cases}$$

maps $|z| = r$ onto a disc centered at $C(r)$ and having the radius $\rho(r)$ given by

$$(2.6) \quad C(r) = \begin{cases} \left(\frac{-B(\gamma - pB)r^2}{1 - B^2r^2}, 0 \right) & \text{for } B \neq 0, \\ (0, 0) & \text{for } B = 0 \end{cases}$$

and

$$(2.7) \quad \rho(r) = \begin{cases} \frac{(\gamma - pB)r^2}{1 - B^2r^2} & \text{for } B \neq 0, \\ |\gamma|r & \text{for } B = 0. \end{cases}$$

Proof. Consider

$$(2.8) \quad \begin{cases} \omega = \frac{(\gamma - pB)z}{1 + Bz} \Leftrightarrow z = \frac{\omega}{\gamma - pB - B\omega} \Leftrightarrow |z|^2 = r^2 = \frac{|\omega|^2}{|\gamma - pB - B\omega|^2} & \text{for } B \neq 0 \\ u^2 + v^2 + \left(\frac{2B(\gamma - pB)r^2}{1 - B^2r^2} \right) u + \frac{(\gamma - pB)^2r^2}{1 - B^2r^2} = 0, & \text{for } B \neq 0. \\ \omega = \gamma z \Leftrightarrow z = \frac{\omega}{\gamma} \Rightarrow |z|^2 = r^2 = \frac{|\omega|^2}{|\gamma|^2} \Rightarrow u^2 + v^2 - \gamma^2r^2 = 0 & \text{for } B = 0. \end{cases}$$

Lemma follows from (2.8). \square

Theorem 2.4. Let $f(z) \in \mathcal{A}_p$ be such that

$$(2.9) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \prec \begin{cases} \frac{(\gamma - pB)z}{1 + Bz} = F_1(z) & \text{for } B \neq 0, \\ \gamma z = F_2(z) & \text{for } B = 0. \end{cases}$$

Then, $f(z) \in \mathcal{S}_q(A, B, p, \alpha)$ and this result is sharp being obtained by the function $\frac{p + \gamma z}{1 + Bz}$.

Proof. Define

$$(2.10) \quad \frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases} (1 + B\omega(z))^{\frac{(\gamma - pB)}{B}} & \text{for } B \neq 0, \\ e^{\gamma\omega(z)} & \text{for } B = 0 \end{cases}$$

where $(1 + B\omega(z))^{\frac{(\gamma - pB)}{B}}$ and $e^{\gamma\omega(z)}$ have the value at 1 at the origin.

Then $\omega(z)$ is analytic in \mathcal{U} and $\omega(0) = 0$. On logarithmic differentiation we

get,

$$(2.11) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \prec \begin{cases} \frac{(\gamma - pB)z\omega'(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\ \gamma z\omega'(z) & \text{for } B = 0. \end{cases}$$

Now it is easy to realize that subordination (2.9) is equivalent to $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. By Jack's Lemma it follows that, there exists a point $z_1 \in \mathcal{U}$ such that

$$(2.12) \quad \frac{z_1 f^{(q+1)}(z_1)}{f^{(q)}(z_1)} - p + q \prec \begin{cases} \frac{(\gamma - pB)k\omega(z_1)}{1 + B\omega(z_1)} = F_1(\omega(z_1)) \notin F_1(\mathcal{U}) & \text{for } B \neq 0, \\ \gamma k\omega(z_1) = F_2(\omega(z_1)) \notin F_2(\mathcal{U}) & \text{for } B = 0. \end{cases}$$

This contradicts our assumption given by (2.9) and the fact that $|\omega(z)| < 1$ for all $z \in \mathcal{U}$.

By using the condition (2.11), we get

$$(2.13) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + q = \begin{cases} \frac{p + \gamma\omega(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\ p + \gamma\omega(z) & \text{for } B = 0. \end{cases}$$

Now by inequality (2.13) we obtain

$$(2.14) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + q \prec \begin{cases} \frac{p + \gamma z}{1 + Bz} & \text{for } B \neq 0, \\ p + \gamma z & \text{for } B = 0. \end{cases}$$

By inequality (2.14) it follows that $f(z) \in \mathcal{S}_q(A, B, p, \alpha)$. □

Corollary 2.5. *Let $f(z) \in \mathcal{S}_q(A, B, p, \alpha)$. Then, $f(z)$ can be written in the form*

$$(2.15) \quad f_*^{(q)}(z) = \begin{cases} z^{p-q}(1 + B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\ z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0. \end{cases}$$

Theorem 2.6. *The radius of starlikeness and the radius of convexity of the $\mathcal{S}_q(A, B, p, \alpha)$ is given by*

$$(2.16) \quad R_{sc} = \frac{2(p-q)}{(\gamma - pB) + \sqrt{(\gamma - pB)^2 + 4(p-q)[(\gamma - pB) + (p-q)B^2]}}.$$

The radius is sharp, being attained by the function

$$(2.17) \quad f_*^{(q)}(z) = \begin{cases} z^{p-q}(1+B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\ z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0. \end{cases}$$

Proof. By Lemma 2.2, the set of values $\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}$ is obtained which comprises the closed disc with center at $C(r)$ and the radius $\rho(r)$, where

(2.18)

$$C(r) = \frac{(p-q) - [B(\gamma-pB) + (p-q)B^2]r^2}{1-B^2r^2} \quad \text{and} \quad \rho(r) = \frac{(\gamma-pB)r}{1-B^2r^2}.$$

Now by the definition of the class $\mathcal{S}_q(A, B, p, \alpha)$ we have,

$$(2.19) \quad \left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r).$$

This gives,

(2.20)

$$\Re \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) \geq \frac{(p-q) - (\gamma-pB)r - [B(\gamma-pB) + (p-q)B^2]r^2}{1-B^2r^2}.$$

Hence for $r < R_{sc}$ the right hand side of the preceding inequality is positive, implying that

$$(2.21) \quad R_{sc} \leq \frac{2(p-q)}{(\gamma-pB) + \sqrt{(\gamma-pB)^2 + 4(p-q)[(\gamma-pB) + (p-q)B^2]}}.$$

The radius is sharp, being attained by the function $f_*^{(q)}(z)$ given by (2.17). \square

Remark 2.7. For parametric values of A, B, p and α we get the well known results proved by Aouf, Nasr and also the results of Yasar Polatoglu.

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